USACH-FM-00/11

Nonlinear Supersymmetry, Quantum Anomaly and Quasi-Exactly Solvable Systems

Sergey M. Klishevich[a,b*] and Mikhail S. Plyushchay[a,b†]

[a] Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile

[b] Institute for High Energy Physics, Protvino, Russia

Abstract

The nonlinear supersymmetry of one-dimensional systems is investigated in the context of the quantum anomaly problem. Any classical supersymmetric system characterized by the nonlinear in the Hamiltonian superalgebra is symplectomorphic to a supersymmetric canonical system with the holomorphic form of the supercharges. Depending on the behaviour of the superpotential, the canonical supersymmetric systems are separated into the three classes. In one of them the parameter specifying the supersymmetry order is subject to some sort of classical quantization, whereas the supersymmetry of another extreme class has a rather fictive nature since its fermion degrees of freedom are decoupled completely by a canonical transformation. The nonlinear supersymmetry with polynomial in momentum supercharges is analysed, and the most general one-parametric Calogero-like solution with the second order supercharges is found. Quantization of the systems of the canonical form reveals the two anomaly-free classes, one of which gives rise naturally to the quasi-exactly solvable systems. The quantum anomaly problem for the Calogero-like models is "cured" by the specific superpotential-dependent term of order $\hbar^2$. The nonlinear supersymmetry admits the generalization to the case of two-dimensional systems.

1 Introduction

After introducing the supersymmetric quantum mechanics as a toy model for studying the supersymmetry breaking mechanism [1, 2], it was applied for solving many problems in theoretical and mathematical physics [3, 4]. The most recent applications of the supersymmetric quantum mechanics can be found in the dynamics of D-branes and black holes [5, 6], in

*E- mail: sklishev@lauca.usach.cl
†E- mail: mplyushc@lauca.usach.cl

M-theory and matrix models [7], in the theory of integrable systems [4] and fluid mechanics [8, 9].

Some time ago it was observed that the pure parabosonic [10] (and parafermionic [11]) systems possess the supersymmetry characterized by the nonlinear superalgebra. Such nonlinear supersymmetry takes place in the simple quantum mechanical system generalizing the usual superoscillator [10], and earlier it was revealed in a similar but particular form in the fermion-monopole system [12, 13], and in the P,T-invariant systems of planar fermions [14] and Chern-Simons fields [15]. The algebraic structure of the nonlinear supersymmetry resembles the structure of the finite W-algebras [16] for which the commutator of any two generating elements is proportional to a finite order polynomial in them.

As it was noted in Ref. [10], in a generic case under attempt of constructing the quantum analogue of the classical systems possessing the nonlinear supersymmetry one faces the problem of the quantum anomaly [17, 18, 19]. To resolve this problem, in the present paper we investigate the nonlinear supersymmetry of one-dimensional systems at the classical and quantum levels. This will allow us to reveal the unexpected very close relation of the nonlinear supersymmetry with associated quantum anomaly problem to the quasy-exactly solvable systems [20, 21, 22, 23, 24] being related, in turn, to the conformal field theory [25, 26, 27, 28, 29]. The results will give also a new perspective on the known quantum supersymmetry characterized by the second order supercharges [30].

The paper is organized as follows. Section 2 is devoted to the detailed investigation of the various aspects of the classical one-dimensional systems possessing the nonlinear supersymmetry of the most general form. In Section 3 we consider their quantization, and reveal the two classes of anomaly-free quantum systems possessing the nonlinear supersymmetry of arbitrary order $k \in \mathbb{Z}_+$. One of them turns out to be closely related to the quasy-exactly solvable systems and this aspect of the nonlinear supersymmetry is investigated in Section 4. Section 5 discusses the general quantum case of the $k = 2$ supersymmetry in the context of anomaly-free quantization of the class of $k = 2$ Calogero-like supersymmetric systems found and analyzed in Section 2. We show also how this $k = 2$ supersymmetry can be used for constructing new exactly solvable systems. In Section 6 the brief summary of the obtained results is presented and some open problems to be interesting for further investigation are discussed. In particular, we point out how the nonlinear supersymmetry can be generalized to the case of the two-dimensional classical and quantum systems.

2 Classical supersymmetry

In this section we investigate the classical supersymmetry of the most general form realizable in one-dimensional boson-fermion system. We shall show that in generic case the supersymmetry is characterized by a nonlinear Poisson algebra and includes the usual supersymmetry as a particular case. Analysing the structure of the supersymmetry from the viewpoint of canonical transformations, we shall observe the existence of the three essentially different classes: in the first class the parameter characterizing the order of superalgebra is subject to the classical quantization, in the second (intermediate) class the supercharges' Poisson bracket can be equal to any real nonnegative degree of the Hamiltonian, whereas the systems of the third class allow ones the complete classical decoupling of the fermion from the boson

degrees of freedom.

2.1 General structure of supersymmetry

Following Ref. [31], let us consider a non-relativistic particle in one dimension described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - V(x) - L(x)N + i\theta^+\dot{\theta}^-, \tag{2.1}$$

where $\theta^\pm$ are the Grassmann variables, $(\theta^+)^* = \theta^-$, $N = \theta^+\theta^-$, and $V(x)$ and $L(x)$ are two real functions. The nontrivial Poisson-Dirac brackets for the system are $\{x,\, p\} = 1$ and $\{\theta^+,\, \theta^-\} = -i$, and the Hamiltonian is

$$H = \frac{1}{2}p^2 + V(x) + L(x)N. \tag{2.2}$$

The latter generates the equations of motion

$$\dot{x} = p, \qquad \dot{p} = -V'(x) - L'(x)N, \qquad \dot{\theta}^\pm = \pm iL(x)\theta^\pm.$$

The Hamiltonian H and the nilpotent quantity N are the even integrals of motion for any choice of the functions $V(x)$, $L(x)$, whereas the odd quantities

$$Q^\pm = B^\mp(x,\, p)\theta^\pm, \quad (B^+)^* = B^-, \tag{2.3}$$

are the integrals of motion when the differential equations

$$\left(p\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial p} \mp iL(x)\right) B^\pm(x,\, p) = 0 \tag{2.4}$$

have the solutions being regular functions in the corresponding domain of the phase space defined by $V(x)$ and $L(x)$. It is obvious that such odd integrals can exist only for a special choice of the functions $V(x)$ and $L(x)$. Let us investigate this question in detail and restrict ourselves to the physically interesting class of the systems given by the potential $V(x)$ bounded from below. Such a potential can generally be represented in terms of a superpotential $W(x)$ and real constant v:

$$V(x) = \frac{1}{2}W^2(x) + v.$$

The condition of regularity of $B^\pm(x,p)$ at $p = 0$ leads to the relation $L(x) = W'(x)\phi(x)$ with some function $\phi(x)$. Having in mind that for the functions $B^\pm(x,\, p)$ the substitution $p \to -p$ is equivalent to the complex conjugation, one can represent them as $B^\pm(x,\, p) = B(W(x),\, \mp ip)$. Then the Hamiltonian and Eq. (2.4) take the form

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v + W'(x)\tilde{\phi}(W)N, \tag{2.5}$$

$$\left(p\frac{\partial}{\partial W} - W\frac{\partial}{\partial p} + i\tilde{\phi}(W)\right) B\,(W,\, ip) = 0, \tag{2.6}$$

where $\tilde{\phi}(W(x)) = \phi(x)$. In terms of the complex variables

$$z = W(x) + ip, \qquad \bar{z} = W(x) - ip \tag{2.7}$$

Eq. (2.6) is represented as

$$\left(\bar{z}\partial_{\bar{z}} - z\partial_z + \tilde{\phi}\left(\tfrac{z+\bar{z}}{2}\right)\right) B(z,\, \bar{z}) = 0. \tag{2.8}$$

General solution to Eq. (2.8) is given by

$$B(\rho, \varphi) = f(\rho) \exp\left(i \int_{\varphi_0}^{\varphi} \tilde{\phi}(\rho\cos\lambda)\, d\lambda\right), \tag{2.9}$$

where $f(\rho)$ is an arbitrary function and $z = \rho e^{i\varphi}$. The simplest case $\phi(x) = \alpha \geq 0$ with $f(\rho) = \rho^{\alpha}$ corresponds to the holomorphic solution of Eq. (2.8), $B(z) = z^{\alpha}$, whereas the case $\alpha \leq 0$ with $f(\rho) = \rho^{-\alpha}$ gives the antiholomorphic solution $B(\bar{z}) = \bar{z}^{-\alpha}$, both regular at $z = \bar{z} = 0$. If the superpotential $W(x)$ is the unbounded function, one of the functions $B(z) = z^{\alpha}$ or $B(\bar{z}) = \bar{z}^{-\alpha}$ is well defined on the whole complex plane only for $\alpha \in \mathbb{Z}$, i.e. we have here some sort of classical quantization [32]. This simplest solution with $\alpha \in \mathbb{Z}$ corresponds to the nonlinear supersymmetry investigated in Ref. [10], and includes the usual linear supersymmetry with $|\alpha| = 1$ as a particular case.

According to the definition (2.3), the functions $B^{\pm}(x,\, p)$ are defined up to an additive nilpotent term proportional to[1] N. This allows ones to represent the supercharges in the equivalent form

$$Q^{\pm} = f(H)e^{\pm i\Phi(p,W(x))}\theta^{\pm}, \quad \Phi(p,\, W(x)) = \int_{\varphi_0}^{\varphi} \tilde{\phi}(\rho\cos\lambda)\, d\lambda. \tag{2.10}$$

We suppose here that like in the case $\phi(x) = \alpha$, the function $f(H)$ is chosen in the simplest form compatible with the requirement of regularity of the supercharges (see below). The supercharges (2.10), the Hamiltonian (2.5) and the nilpotent integral N form generally the nonlinear superalgebra

$$\left\{Q^{-},\, Q^{+}\right\} = -if^2(H), \quad \{N, Q^{\pm}\} = \mp iQ^{\pm}, \quad \{Q^{\pm}, H\} = 0, \quad \{N, H\} = 0.$$

2.2 Three cases of supersymmetry and classical quantization

In the case of unbounded superpotential $W(x)$, the dynamics of the system projected on the unity of the Grassmann algebra is defined on the whole complex plane $\mathbb{C}$. In order to have the supercharges (2.10) to be well defined single-valued observables on $\mathbb{C}$, we have to impose the condition

$$\int_0^{2\pi} \tilde{\phi}\,(\rho\cos\lambda)\; d\lambda = 2\pi k, \qquad k \in \mathbb{Z}.$$

[1] At the quantum level the terms $N\theta^{\pm}$, N^2 etc. do not vanish and may be essential for preserving the supersymmetry.

The function $\tilde{\phi}$ is supposed to be regular and can be represented in the form

$$\tilde{\phi}(W(x)) = k + W(x)M(W^2(x)), \tag{2.11}$$

where $M(W^2)$ is an arbitrary function, $|M(0)| < \infty$. As a consequence, the supercharge can be written as

$$Q^+ = z^k e^{i\int_0^p M(p^2-y^2+W^2(x))\,dy}\theta^+. \tag{2.12}$$

Here we suppose that k is nonnegative; in the case of negative k equation (2.12) with substitutions $k \to -k$, $\theta^+ \to \theta^-$ gives the supercharge $Q^- = (Q^+)^*$. The exponential factor in Eq. (2.12) can be removed by the transformation

$$\Theta^\pm = e^{\pm iG(p,\,x)}\theta^\pm, \qquad X = x + \partial_p G(p,x)N, \qquad P = p - \partial_x G(p,x)N, \tag{2.13}$$

that is the canonical transformation with generating function G obeying the differentiability condition

$$\partial_p\partial_x G(p,x) = \partial_x\partial_p G(p,x).$$

In the case of general superpotential $W(x)$, the transformation (2.13) can be used to reduce the classical Hamiltonian (2.5) to the most simple form, e.g., to the form with $L(x) = \alpha W'(x)$, $\alpha = const$. This gives the following equation for the function G:

$$\left(p\frac{\partial}{\partial x} - W(x)W'(x)\frac{\partial}{\partial p}\right) G(x,\,p) + W'(x)\left(\tilde{\phi}(W(x)) - \alpha\right) = 0. \tag{2.14}$$

Though the equation (2.14) has exactly the form of that for $\log B$ with shifted $\tilde{\phi}$ (see Eq. (2.6)), there is a difference: we require for G to be regular and single-valued function on the whole physical domain, whereas the same condition of regularity is imposed on the function B, but not on $\log B$. The general solution to Eq. (2.14) is

$$G(p,\,x) = \int_{\varphi_0}^{\varphi}\left(\tilde{\phi}(\rho\cos\lambda) - \alpha\right)\,d\lambda,$$

and its behaviour depends on the physical domain for z defined, in turn, by the properties of the superpotential $W(x)$. Here it is necessary to separate the three different cases and the results can be summarized as follows.

1. The physical domain in terms of z includes the origin ($a < W(x) < b$, $a < 0$, $b > 0$). This, in particular, corresponds to the case of unbounded superpotential with $a = -\infty$, $b = +\infty$. From the regularity of G in such a domain it follows that $\alpha = \tilde{\phi}(0)$. We assume that the function $\tilde{\phi}(W)$ can be decomposed into the Taylor series at $W = 0$. From the regularity of B we arrive at the classical "quantization" condition $\alpha = k$, $k \in \mathbb{Z}$, and at the restriction (2.11) on the function $\tilde{\phi}(W)$. Thus, the most general form of the Hamiltonian admitting the nonlinear supersymmetry is

$$H = \frac{p^2}{2} + \frac{1}{2}W^2(x) + v + W'(x)\left[k + W(x)M(W^2(x))\right]N, \qquad k \in \mathbb{Z}, \tag{2.15}$$

whose associated supercharges have been described above. By the canonical transformation (2.13) with $G(p,\, x) = \int_0^p M(p^2 - y^2 + W^2(x))\, dy$ (supplemented by the transformation $\Theta^\pm \to \Theta^\mp$ in the case $k < 0$) we can always reduce the system with this Hamiltonian and the supercharge (2.12) to the form of the supersymmetric system possessing the holomorphic supercharge:

$$H = \frac{1}{2}P^2 + \frac{1}{2}W^2(X) + v + kW'(X)\Theta^+\Theta^-, \qquad Q^+ = Z^k\Theta^+, \qquad k \in \mathbb{Z}_+, \tag{2.16}$$

where $Z = W(X) + iP$. The presence of the "quantized", integer number k in the Hamiltonian (2.16) means that the instant frequencies of the oscillator-like odd, $\Theta^\pm$, and even, Z, $\bar{Z}$, variables are commensurable. Only in this case the regular odd integrals of motion can be constructed, and the factor Z^k in the supercharge Q^+ corresponds to the k-fold conformal mapping of the complex plane (or the strip $a < \mathrm{Re}\, \mathrm{Z} < \mathrm{b}$) on itself (or on the corresponding region in $\mathbb{C}$).

2. The physical domain is defined by the condition $\mathrm{Re}\, z \geq 0$ (or $\mathrm{Re}\, z \leq 0$) and also includes the origin of the complex plane. But unlike the previous case, there are no closed contours around $z = 0$. As a consequence, though the regularity of G results in the same relation $\alpha = \tilde{\phi}(0)$, no "quantization" condition appears from the regularity of B. The most general form of the Hamiltonian and the supercharge is

$$H = \frac{p^2}{2} + \frac{1}{2}W^2(x) + v + W'(x)\left[\alpha + R(W(x))\right] N, \quad Q^+ = z^\alpha e^{i\int_{\varphi_0}^{\varphi} R(\rho\cos\lambda)\, d\lambda}\theta^+, \tag{2.17}$$

where we assume that $\alpha \in \mathbb{R}$, and the function $R(W)$ is analytical at $W = 0$ and $R(0) = 0$. The singularity in Q^+ at the origin $z = 0$ for $\alpha < 0$ is not physical and can be removed multiplying Q^+ by $(\bar{z}z)^{-\alpha}$, that results in changing the holomorphic function $B(z, \bar{z}) = z^\alpha$ for the antiholomorphic function $B(z, \bar{z}) = \bar{z}^{-\alpha}$ to be regular at $\bar{z} = 0$. After the canonical transformation (2.13) with the function $G(p,\, x) = \int_{\varphi_0}^{\varphi} R(\rho\cos\lambda)\, d\lambda$ (supplemented by the transformation $\Theta^\pm \to \Theta^\mp$ for $\alpha < 0$), the Hamiltonian and the supercharge can be reduced to the form

$$H = \frac{1}{2}P^2 + \frac{1}{2}W^2(X) + v + \alpha W'(X)\Theta^+\Theta^-, \quad Q^+ = Z^\alpha\Theta^+, \quad \alpha \in \mathbb{R}_+. \tag{2.18}$$

3. The physical domain is defined by the condition $\mathrm{Re}\, z > 0$ (or $\mathrm{Re}\, z < 0$), i.e. the origin of the complex plane is not included. In this case α and $\tilde{\phi}(0)$ are not related since the function G admits in such a domain the terms proportional to $\arg z$. Therefore, though the general form of the Hamiltonian and the supercharge is

$$H = \frac{p^2}{2} + \frac{1}{2}W^2 + v + W'\tilde{\phi}(W)N, \qquad Q^+ = f(H)e^{i\int_{\varphi_0}^{\varphi} \tilde{\phi}(\rho\cos\lambda)\, d\lambda}\theta^+, \tag{2.19}$$

by the canonical transformations (2.13) with the function $G(p,\, x) = \int_{\varphi_0}^{\varphi} \tilde{\phi}(\rho\cos\lambda)\, d\lambda$, one can reduce the Hamiltonian to the form

$$H = \frac{1}{2}P^2 + \frac{1}{2}W^2(X) + v \tag{2.20}$$

with trivial dynamics for the Grassmann variables $\Theta^{\pm}$, $\dot{\Theta}^{\pm}=0$, playing the role of the supercharges. Thus, *classically* the supersymmetry of any system with bounded non-vanishing superpotential has a "fictive" nature.

The obtained classification of the classical supersymmetric systems emerged from the aim to present the Hamiltonian in the most simple form when the superpotential $W(x)$ has the definite behaviour defining the type of the physical domain in the complex plane. On the other hand, one can consider the supersymmetry from the viewpoint of the functional dependence of the Hamiltonian on $W(x)$ without specifying the superpotential's type. Then the Hamiltonian and the supercharges

$$H=\frac{1}{2}p^2+\frac{1}{2}W^2(x)+v+kW'(x)\theta^+\theta^-,\quad Q^+=z^k\theta^+,\quad Q^-=\bar{z}^k\theta^-,\quad k\in\mathbb{Z}_+ \tag{2.21}$$

give the intersection of the described three classes of the systems, and can be treated as the representatives of the more broad classes of the supersymmetric systems (2.15), (2.17) and (2.19), with which (2.21) is related by the corresponding symplectomorphism. For (2.21) the constant k characterizes the degree of nonlinearity of the associated superalgebra and one can refer to it as to the system with k-supersymmetry. At the same time, it is necessary to bare in mind that in the case of the superpotential of the third class all the systems (2.21) with different $k\neq 0$ are symplectomorphic to the system with $k=0$.

The three types of supersymmetric Hamiltonians (2.15), (2.17), (2.18) are defined up to the additive constant v. This arbitrariness could be used to present the potential $V(x)$ in terms of other superpotential via the relation

$$\tilde{W}^2(x)=W^2(x)+const. \tag{2.22}$$

Generally, the superpotentials $W(x)$ and $\tilde{W}(x)$ can correspond to different types of nonlinear supersymmetry. Then the natural question is: can such a transition change the type of supersymmetry when the form of the nilpotent term of the Hamiltonian has been already fixed? In other words, the question is if the described classification of *classical* supersymmetric Hamiltonians has an invariant sense. The invariance of the classification is demonstrated in Appendix. At the same time, it is necessary to stress that the systems related by the canonical transformation have not to be equivalent on the quantum level due to the ordering problem and because the canonical transformations are, as a rule, nonpolynomial in momenta.

For the sake of completness, let us discuss the Lagrangian formulation for the k-supersymmetric system (2.21). Its Lagrangian is

$$L=\frac{1}{2}\dot{x}^2-\frac{1}{2}W^2(x)-v-kW'(x)\theta^+\theta^-+i\theta^+\dot{\theta}^-. \tag{2.23}$$

In the Hamiltonian formulation the supertransformations of the variables x and $\theta^{\pm}$ are generated canonically by the supercharges:

$$\delta x=\{x,\,Q^+\}\eta^- - \{x,\,Q^-\}\eta^+ = ik\left[z^{k-1}\theta^+\eta^- + \bar{z}^{k-1}\theta^-\eta^+\right],$$
$$\delta\theta^+=\{\theta^+,\,Q^-\}\eta^+ = -iz^k\eta^+,\quad \delta\theta^- = -\{\theta^-,\,Q^+\}\eta^- = i\bar{z}^k\eta^-.$$

Using the equations of the motion, we obtain the corresponding supertransformations at the Lagrangian level:

$$\delta x = ik\left[\left(A^-\right)^{k-1}\theta^+\eta^- + \left(A^+\right)^{k-1}\theta^-\eta^+\right], \qquad \delta\theta^\pm = \mp i\left(A^\pm\right)^k\eta^\pm,$$

where $A^\pm = W(x) \mp i\dot{x}$. The Lagrangian (2.23) is quasi-invariant under these supertransformations:

$$\delta L = \frac{d}{dt}\left[ik\dot{x}\Big((A^-)^{k-1}\theta^+\eta^- + (A^+)^{k-1}\theta^-\eta^+\Big)\right].$$

It is worth noting that on shell the commutator of the two supertransformations for the physical variables is proportional to the translation in time:

$$[\delta_1, \delta_2]\mathcal{X} = -i2^k k E^{k-1}(x, \dot{x}, \theta^\pm)(\eta_1^-\eta_2^+ - \eta_2^-\eta_1^+)\frac{d}{dt}\mathcal{X},$$

where $\mathcal{X}$ is x or $\theta^\pm$, and $E(x, \dot{x}, \theta^\pm)$ is the energy function of the system. Therefore, the supertransformations form an open algebra with the structure functions depending on the physical variables.

2.3 Supersymmetry with polynomial supercharges

We have arrived at the three types of Hamiltonians (2.15), (2.17), (2.18), which after appropriate canonical transformations can be reduced to the more simple form with the associated supercharges represented in the holomorphic or antiholomorphic form. On the other hand, as it was mentioned above, the quantization of canonically equivalent classical systems can give in some cases the quantum systems with different types of supersymmetry, even of different order k. Therefore, the search for other special representations for the Hamiltonian and associated supercharges is important. Based on this remark, let us look for the representation in which the function $B(x,\, ip)$ defining the supercharges is the polynomial in p of the degree k, i.e.

$$B(x,\, ip) = \sum_{n=0}^{k} b_{k-n}(x)(ip)^n.$$

Substituting this into Eq. (2.4), we obtain the recurrent equation

$$b_n'(x) + (k-n+2)\, b_{n-2}(x)V'(x) - L(x)b_{n-1}(x) = 0, \qquad (2.24)$$

where $b_n(x) = 0$ for $n < 0$ and $n > k$ is assumed. Due to the equation $b_0'(x) = 0$, one can fix $b_0(x) = 1$. Then the part of the equations (2.24) can be solved giving for $L(x)$ and $V(x)$ the relations

$$L(x) = b_1'(x), \qquad V(x) = \frac{1}{k}\left(\frac{b_1(x)^2}{2} - b_2(x)\right) + v.$$

To simplify the notation, we put $b_1(x) = y(x)$ and realize the change of the variables

$$x = x(y), \qquad \varphi_n(y) = b_n(x(y)), \quad \text{for} \quad n > 1. \qquad (2.25)$$

If the function inverse to $y(x)$ does not exist globally, we can perform this transformation separately on each interval where the function $y(x)$ is monotonic. Under the transformation (2.25), Eq. (2.24) acquires the form of the system of differential equations in the variable y:

$$\varphi_n'(y) - \frac{k-n+2}{k}\varphi_{n-2}(y)\left(\varphi_2'(y) - y\right) - \varphi_{n-1}(y) = 0. \tag{2.26}$$

A general solution to this system has $k-1$ real parameters. If we know solution to the system (2.26), we could find the form of the functions $b_n(x)$, at least implicitly. This means that the general solution to Eq. (2.24) depends on arbitrary function $b_1(x)$ but not on its derivatives, and which can be called the superpotential. One notes also that the system (2.26) is invariant under the transformation $y \to -y$ if the functions φ_n obey the relation

$$\varphi_n\left(-y\right) = (-1)^n\, \varphi_n\left(y\right). \tag{2.27}$$

The solution corresponding to the holomorphic case is of this type.

The simplest case $k=1$ corresponds to the usual linear supersymmetry, and we turn to the case $k=2$. For $k=2$, from (2.26) we obtain the equation for φ_2:

$$y\varphi_2' + 2\varphi_2 = y^2.$$

This equation has the solution

$$\varphi_2 = \frac{y^2}{4} + \frac{4c}{y^2},$$

where c is an arbitrary real constant. Considering $W(x) = y(x)/2$ as a superpotential, one arrives at the supercharges of the form

$$Q^{\pm} = \frac{1}{2}\left[\left(\pm ip + W(x)\right)^2 + \frac{c}{W^2(x)}\right]\theta^{\pm}, \tag{2.28}$$

which together with the Hamiltonian

$$H = \frac{1}{2}\left[p^2 + W^2(x) - \frac{c}{W^2(x)}\right] + 2W'(x)N + v \tag{2.29}$$

form the nonlinear superalgebra

$$\left\{Q^-, Q^+\right\} = -i\left((H-v)^2 + c\right), \qquad \left\{Q^{\pm}, H\right\} = 0. \tag{2.30}$$

Note that the Hamiltonian (2.29) has the Calogero-like form: at $W(x) = x$ its projection to the unit of Grassmann algebra takes the form of the Hamiltonian of the two-particle Calogero system. The functions $B^{\pm}(p,\, x) = B(x, \mp ip)$ and Hamiltonian (2.29) form the nonlinear Poisson algebra

$$\left\{B^{\pm},\, H\right\} = \pm iW'B^{\pm}, \qquad \left\{B^-,\, B^+\right\} = -4iW'H,$$

which does not depend on the constant c, and is reduced to the $sl(2,R)$ algebra at $W(x) = x$.

For $c = 0$ the obtained $k = 2$ supersymmetric system (2.28), (2.29) is reduced to the $k = 2$ supersymmetric system of the form (2.21) characterized by the holomorphic form of the supercharge. Let us investigate the relationship of the system (2.28), (2.29) with k-supersymmetry (2.21) in the case $c \neq 0$. First we show that for $c < 0$ ($c = -\gamma^2$, $\gamma > 0$) the system (2.29), (2.28) can be reduced to the linear ($k = 1$) supersymmetric system of the form (2.21) with the corresponding holomorphic supercharge multiplied by some function of the Hamiltonian. To show this we represent the potential of the Hamiltonian (2.29) in the form

$$V(x) = \frac{1}{2}\left(W(x) - \frac{\gamma}{W(x)}\right)^2 + v + \gamma = \frac{1}{2}\tilde{W}^2(x) + v + \gamma,$$

Without loss of generality we can consider $W(x) > 0$ and get the relation

$$W(x) = \frac{1}{2}\left(\tilde{W}(x) + \sqrt{\tilde{W}^2(x) + 4\gamma}\right).$$

Therefore, in terms of $\tilde{W}(x)$ the function $L(x)$ is represented as

$$L(x) = \tilde{W}'(x)\left[1 + \tilde{W}(x)M(\tilde{W}^2(x))\right], \quad M(\tilde{W}^2(x)) = \left(\tilde{W}^2(x) + 4\gamma\right)^{-1/2}.$$

Comparing this with the factor at the nilpotent term in Hamiltonians (2.15), (2.17) and (2.18), we find that our system corresponds to the system (2.21) with $k = 1$ since the term with function $M(\tilde{W}^2(x))$ can be removed by the appropriate canonical transformation. Therefore, we conclude that the system (2.29) is canonically equivalent to the $k = 1$ supersymmetric system (2.21).

Analogously, applying the canonical transformation (2.13) to the system (2.29) with $c > 0$ ($c = \gamma^2$, $\gamma \neq 0$), one can reduce it to the $k = 0$ supersymmetric system with the Hamiltonian of the form (2.20). Indeed, let us define the new superpotential $\tilde{W}(x)$ by the relation

$$\tilde{W}^2(x) + 2\tilde{v} = W^2(x) - \frac{\gamma^2}{W^2(x)} + 2v = 2V(x),$$

where without loss of generality we assume that $W(x) > 0$, and the constant $\tilde{v}$ has to be chosen to provide the inequality $\tilde{W}^2(x) \geq 0$. As a result, one can represent $W(x)$ in terms of the new superpotential as

$$W(x) = 2^{-\frac{1}{2}}\sqrt{\mathcal{D} + \sqrt{\mathcal{D}^2 + 4\gamma^2}}, \qquad \mathcal{D} = \tilde{W}^2(x) - 2\Delta v,$$

where $\Delta v = v - \tilde{v}$. Expressing $L(x)$ in terms of $\tilde{W}$, we obtain $L(x) \propto \tilde{W}'(x)\tilde{W}(x)M(\tilde{W}^2(x))$, with the function $M(\tilde{W}^2(x)) = [1 - [\mathcal{D}^2 + 4\gamma^2]^{-1/2}] \cdot W^{-\frac{1}{2}}(\tilde{W}(x))$ to be regular due to the inequality $W(x) > 0$. Therefore the nilpotent term can be removed from the Hamiltonian by the canonical transformation[2] (2.13) since the generating function $G(p, \tilde{W})$ is regular in this case.

[2]See Appendix for the details.

Let us turn now to the next, $k=3$ case given by the system of equations

$$\varphi_3' - \frac{2}{3}y\left(\varphi_2' - y\right) - \varphi_2 = 0, \quad \varphi_3 + \frac{1}{3}\varphi_2\left(\varphi_2' - y\right) = 0. \tag{2.31}$$

Substituting φ_3 from the second equation into the first one, we arrive at the nonlinear differential equation of the second order

$$\varphi_2\varphi_2'' + \left(\varphi_2'\right)^2 + y\varphi_2' + 2\varphi_2 - 2y^2 = 0,$$

which can equivalently be represented as the first order nonlinear differential equation

$$y\varphi_2\varphi_2' + \frac{1}{2}\left(y^2 - \varphi_2\right)\left(y^2 + \varphi_2\right) + y^2\left(\varphi_2 - y^2\right) - C = 0.$$

When the integration constant $C=0$, the solution to the equation is a root of the 4th order algebraic equation for $\varphi_2(y^2)$:

$$\left(3\varphi_2 - y^2\right)\left(y^2 + \varphi_2\right)^3 - C_1 y^2 = 0.$$

At $C_1 = 0$, its solutions have a simple form: $\varphi_2 = y^2/3$, $\varphi_2 = -y^2$. The first solution corresponds to the holomorphic case with $k=3$, while the second one does to the case of $k=1$ supersymmetry with the supercharge multiplied by the Hamiltonian. Even in the case $C=0$, $C_1 \neq 0$, the corresponding solutions $\varphi_2(y^2)$ have a complicated form of solutions in radicals of the 4th order equation, whereas for $C \neq 0$ we have not succeeded in finding any analytical solution of the nontrivial nature. The same complications appear with finding the nontrivial solutions to the system (2.26) for $k>3$.

3 Quantum anomaly and k-supersymmetry

As we have seen in the previous section, the supersymmetry in classical one-dimensional system is defined by the arbitrary function $W(x)$ and in general case the supercharges together with the Hamiltonian form a nonlinear superalgebra. According to the results of Ref. [10] on the supersymmetry in pure parabosonic systems, a priori one can not exclude the situation characterized by the supercharges to be the nonlocal operators represented in the form of some infinite series in the operator $\frac{d}{dx}$. Since such nonlocal supercharges have to anticommute for some function of the Hamiltonian being a usual local differential operator of the second order, they have to possess a very peculiar structure[3]. Due to this reason, we restrict ourselves by the discussion of the supersymmetric systems with the supercharges being the differential operators of order k. Classically this corresponds to the system (2.21) with the holomorphic supercharges or to the systems discussed in Section 2.2.

In Ref. [10] it was observed that just in the simplest case of the superoscillator possessing the nonlinear k-supersymmetry and characterized by the holomorphic supercharges of the form (2.21) with the simplest superpotential $W(x)=x$, the form of the classical superalgebra $\{Q_k^+, Q_k^-\} = H^k$ is changed for $\{Q_k^+, Q_k^-\} = H(H-\hbar)(H-2\hbar)\dots(H-\hbar(k-1))$ due to

[3]In pure parabosonic systems revealing k-supersymmetry both the supercharges and Hamiltonian have a nonlocal structure [10].

the quantum noncommutativity. Moreover, it was also observed that for $W(x) \neq ax + b$ in generic case we have a global quantum anomaly [17, 18]: the direct quantum analogue of the superoscillators loose the property of the conservation, $[Q^{\pm}_k, H_k] \neq 0$. Therefore, we arrive at the problem of looking for the classes of superpotentials and corresponding quantization prescriptions leading to the quantum k-supersymmetric systems without quantum anomaly. This and next two sections are devoted to the solution of such a problem.

3.1 The k-supersymmetry: quadratic superpotential.

In this and next subsections we consider the quantization of the nonlinear supersymmetry characterized by the holomorphic form of the supercharges (2.21). As we shall see, the straightforward quantization without special quantum corrections leads to the rigid restrictions on the form of the superpotential $W(x)$.

Let us fix the quantum supercharges in the holomorphic form corresponding to the classical k-supersymmetry,

$$Q^{\pm} = 2^{-\frac{k}{2}}(A^{\mp})^k\theta^{\pm}, \tag{3.1}$$

with

$$A^{\pm} = \mp\hbar\frac{d}{dx} + W(x). \tag{3.2}$$

With realization

$$N = \theta^+\theta^- = \frac{\hbar}{2}(\sigma_3 + 1), \tag{3.3}$$

the operator N has a sense of the fermionic quantum number and being normalized for $\hbar$ is the projector onto the fermionic subspace, whereas the complimentary projector onto the bosonic subspace is $\theta^-\theta^+/\hbar = 1 - N\hbar^{-1}$. Then choosing the quantum Hamiltonian in the form (2.2), from the requirement of conservation of the supercharges, $[Q^{\pm}, H] = 0$, we arrive at the equations

$$L(x) = kW'(x), \tag{3.4}$$
$$V(x) = \tfrac{1}{2}\left(W^2(x) - k\hbar W'(x)\right) + v, \tag{3.5}$$
$$\hbar^3 k(k^2-1)W'''(x) = 0. \tag{3.6}$$

Therefore, the quantum system given by the Hamiltonian

$$H = \frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2} + W^2(x) + 2v + k\hbar\sigma_3 W'\right) \tag{3.7}$$

possesses the nonlinear supersymmetry of order $k \geq 2$ characterized by the holomorphic supercharges (3.1) only when

$$W(x) = w_2x^2 + w_1x + w_0. \tag{3.8}$$

For any other form of the superpotential the nilpotent operators (3.1) are not conserved that can be treated as a quantum anomaly. Below we shall see that the quantum anomaly for the system (3.7) can be "cured" for some superpotentials if to modify appropriately (by $\hbar$-dependent terms) the supercharges. It is necessary to stress that the relation (3.6) fixing the form of the superpotential for $k \geq 2$ has a purely quantum nature. One also notes that with the prescription (3.3), $V(x)$ given by Eq. (3.5) plays the role of the total potential for the bosonic sector, $V(x) = V_B(x)$, whereas the potential for the fermionic sector is $V_F(x) = V(x) + \hbar k W'(x)$. The appearance in $V(x)$ of the term proportional to $\hbar$ can obviously be associated with the ordering ambiguity under construction of the operator N: under the choice $N = \frac{1}{2}[\theta^+, \theta^-]$ instead of (3.3), we again arrive at the Hamiltonian (3.7), but the term linear in $\hbar$ dissappears from $V(x)$. In what follows we shall have in mind the quantum prescription (3.3).

The anticommutator of supercharges Q^+ and Q^- gives the polynomial of order k in H, e.g., for the simplest cases $k = 2$, $k = 3$ and $k = 4$ we have

$$\{Q^-, Q^+\} = H^2 - \tfrac{1}{4}\left(w_1^2 - 4w_0 w_2\right),$$
$$\{Q^-, Q^+\} = H^3 - \left(w_1^2 - 4w_0 w_2\right) H + 2w_2^2,$$
$$\{Q^-, Q^+\} = H^4 - \tfrac{5}{2}\left(w_1^2 - 4w_0 w_2\right) H^2 + 12 w_2^2 H + \tfrac{9}{16} w_2^2 \left(w_1^2 - 4w_0 w_2\right)^2,$$

where for the sake of simplicity we have put $v = 0$ and $\hbar = 1$. The family of supersymmetric systems (3.8) is reduced to the superoscillator at $w_2 = 0$ with the associated exact k-supersymmetry [10]. For $w_2 \neq 0$ the k-supersymmetry is realized always in the spontaneously broken phase since in this case the supercharges (3.1) have no zero modes (normalized eigenfunctions of zero eigenvalue).

It is worth noting that the nonlinear supersymmetry with quadratic superpotential (3.8) was found earlier in ref. [33] as a by-product in the context of the discussion of the nonrenormalization theorem for supersymmetric theories.

3.2 The k-supersymmetry: exponential superpotential

Let us take the supercharges in the form of polynomials of order k in the oscillator-like variables $A^\pm$ (3.2):

$$Q^\pm = 2^{-\frac{k}{2}} \left\{ (A^\mp)^k + \sum_{n=0}^{k-1} q_{k-n} (A^\mp)^n \right\} \theta^\pm, \tag{3.9}$$

where q_n are real parameters have to be fixed. If we treat the supercharges (3.9) classically, then the condition of their conservation by the Hamiltonian of the general form (2.2) results in

$$Q^\pm = 2^{\frac{k}{2}} \left(A^\mp + \frac{q_1}{k} \right)^k \theta^\pm, \quad H = \left(\{Q^+, Q^-\} \right)^{\frac{1}{k}} + v,$$

i.e. we arrive again at the k-supersymmetric system (2.21) with the arbitrary function $W(x) + q_1/k$ playing the role of the superpotential. However, as we shall see, quantum mechanically ansatz (3.9) gives us a nontrivial family of k-supersymmetric systems related to the so called quasi-exactly solvable problems [20, 21, 22, 23, 24]. First one notes that the

parameter q_1 can be removed by a simple shift of the superpotential both on the classical and quantum levels and we can put $q_1 = 0$. Then, as in the case of the supercharges (3.1), the requirement of conservation of (3.9) results in the Hamiltonian (3.7) as well as in some $k-2$ algebraic equations fixing the parameters $q_3, q_4, \ldots, q_k$ in terms of $q \equiv q_2$, whereas the condition (3.6) for $k \geq 2$ is changed now for

$$\hbar^3 W''' - \omega_k^2 \hbar W' = 0, \qquad \omega_k^2 = -\frac{24q}{k\,(k^2-1)}. \tag{3.10}$$

This means that the supercharges (3.9) contain only the one-parameter arbitrariness and the case of quadratic superpotential (3.8) is included here as a particular case corresponding to $q = 0$. For $q \neq 0$ the solution of Eq. (3.10) acquires the exponential form

$$W(x) = w_+ e^{\omega_k x} + w_- e^{-\omega_k x} + w_0, \tag{3.11}$$

where all the parameters $w_{\pm,0}$ are real, while the parameter ω_k is real or pure imaginary depending on the sign of q, and for the sake of simplicity we put $\hbar = 1$. In the limit $\omega_k \to 0$ this superpotential is reduced to the quadratic form (3.8) via the appropriate rescaling of the parameters $w_{\pm,0}$.

For $k = 2$ the anticommutator of the supercharges has the form

$$\left\{Q^-, Q^+\right\} = (H - c_-)\,(H - c_+)\,,$$

where

$$c_\pm = \,\pm\,\omega_2 \sqrt{w_0^2 - 4 w_+ w_-} + q + v.$$

For $k \geq 2$ the superalgebra for the system with superpotential (3.11) can be represented as

$$\left\{Q^-, Q^+\right\} = H^k + \sum_{n=1}^{k} c_n H^{k-n}, \qquad c_n \in \mathbb{R}. \tag{3.12}$$

In principle, the coefficients c_n can be found explicitly since the system of linear equations arises for them. The values of energy of the supercharges' singlets are the roots of the polynomial on the right hand side of Eq. (3.12) and to find them one has to solve the algebraic equation of the corresponding order. In next section we analyse in detail the class of k- supersymmetric systems given by the superpotential (3.11) in the context of the partial algebraization of the spectral problem.

4 The k-supersymmetry and partial algebraization of the spectral problem

In the late eighties a new class of spectral problems was discovered [20, 21]. It occupies the intermediate position between the exactly solvable problems and all others. According to Ref. [34], the quantum-mechanical system is quasi-exactly solvable or admits partial algebraization of the spectrum if its potential depends explicitly on the natural parameter n

in such a way that exactly n levels in the spectrum can be found algebraically. The unique nature of such systems is based on the hidden symmetry of the Hamiltonian. The part of the spectrum that can be found algebraically is related to finite-dimensional representations of the corresponding Lee group (algebra). For one-dimensional systems the quasi-exact solvability is associated with non-unitary finite-dimensional representations of the $sl(2,\mathbb{R})$. Such representations are characterized by a parameter j referred to as a "spin", which can take integer and half-integer values. The number of the eigenstates of the Hamiltonian that can be found algebraically is equal to $2j+1$.

Here we argue in favour of existence of the intimate relation between nonlinear supersymmetry and the partial algebraization scheme [21, 34]. For example, if in a given system with the nonlinear supersymmetry of the order k,

$$\{Q^-,Q^+\} = (H-E_1)\ldots(H-E_k)\,,$$

there are k singlets in the bosonic or fermionic sectors, i.e. k zero modes of Q^+ or Q^-, then the eigenvalues of the corresponding states are equal to E_i. Then it is quite obvious that if such a system is not exactly-solvable, it admits the partial algebraization of its spectrum. Having in mind these preliminary comments, let us show that the supersymmetric system with the superpotential (3.11) can be related to some known families of quasi-exactly solvable problems.

Let us put $\hbar = 1$ and consider the potentials

$$V(x) = \frac{1}{2}\left(ae^x+b-2j\right)^2+\frac{1}{2}\left(ce^{-x}+b+1\right)^2, \tag{4.1}$$

$$V(x) = \frac{a^2}{2}\cos^2 x + a\left(2j+\tfrac{1}{2}\right)\sin x, \tag{4.2}$$

$$V(x) = \frac{a^2}{2}\sinh^2 x - a\left(2j+\tfrac{1}{2}\right)\cosh x, \tag{4.3}$$

$$V(x) = \frac{a^2}{2}\cosh^4 x - \frac{a}{2}\left(a+4j+2\right)\cosh^2 x, \tag{4.4}$$

admitting the partial algebraization of the spectrum [21, 34]. The potential (3.5) with the superpotential of the from (3.11) coincides with the potential (4.1) when

$$\omega_k = 1, \quad w_+ = a, \quad w_- = c, \quad w_0 = 1+2\left(b-j\right), \quad k = 2j+1,$$
$$v = \frac{3}{4}+b+b^2-2ac+j-2bj+3j^2,$$

or with the potential (4.2) when

$$\omega_k = -i, \quad w_- = w_+ = \frac{a}{2}, \quad w_0 = 0, \quad v = 0, \quad k = 4j+1, \tag{4.5}$$

or with the potential (4.3) when

$$\omega_k = 1, \quad w_\pm = \pm\frac{a}{2}, \quad w_0 = 0, \quad v = 0, \quad k = 4j+1, \tag{4.6}$$

or with the potential (4.3) when

$$\omega_k = 2, \quad w_\pm = \pm\frac{a}{4}, \quad w_0 = 0, \quad v = -a\left(j + \tfrac{1}{2}\right), \quad k = 2j + 1.$$

In the cases (4.5) and (4.6), there is a complete correspondence with the quasi-exactly solvable potentials (4.2) and (4.3) for odd k only. Having in mind this observation, we first analyse in detail the correspondence between the nonlinear supersymmetry and the partial algebraization scheme for the case of quasi-exactly solvable potential (4.3). Writing down this potential as

$$V(x) = \frac{a^2}{2}\sinh^2 x - \frac{a}{2}k\cosh x, \qquad k \in \mathbb{N}, \tag{4.7}$$

we see that it has exactly the form of the potential (3.5) of the nonlinear supersymmetry system with the corresponding superpotential

$$W(x) = a\sinh x. \tag{4.8}$$

Since the potential of the form (4.7) corresponds to the potential (4.3) for odd k only, let us investigate the k-supersymmetric systems with even k and start from the case $k = 2$. Formally this corresponds to the "spin" $j = 1/4$ in the partial algebraization scheme [21] for the potential (4.3). In this case $q = -1/4$ and the supercharges read as

$$Q^\pm = \frac{1}{2}\left((A^\mp)^2 - \frac{1}{4}\right)\theta^\pm,$$

where $A^\pm$ are defined in (3.2). Zero modes of the supercharge Q^+ belong to the bosonic sector and have the form

$$\varphi_\pm(x) = e^{\pm\frac{x}{2}}e^{-a\cosh x},$$

where for definiteness we have supposed that $a > 0$. The anticommutator of the supercharges is

$$\{Q^-, Q^+\} = \left(H - \frac{a}{2} + \frac{1}{8}\right)\left(H + \frac{a}{2} + \frac{1}{8}\right),$$

and we find that the linear combinations $\varphi^{(\pm)} = \varphi_- \pm \varphi_+$ are the eigenfunctions of the bosonic Hamiltonian,

$$H_B\varphi = E\varphi, \tag{4.9}$$

corresponding to the eigenvalues

$$E_\pm = \mp\frac{a}{2} - \frac{1}{8}.$$

The fermionic Hamiltonian H_F is characterized by the potential

$$V_F(x) = \frac{a^2}{2}\sinh^2 x + \frac{a}{2}k\cosh x,$$

which is the superpartner of the potential (4.7). Since this differs from (4.7) in the sign before the second term, the corresponding solutions in the fermionic sector in the case of $a < 0$ can be obtained from the bosonic solutions with $a > 0$ by a simple change $a \to -a$. Therefore, we see that there are two bound states in the bosonic (fermionic) sector for $a > 0$ ($a < 0$), that means that the potential (4.7) is quasi-exactly solvable for $k = 2$ as well. As we shall see, the same conclusion is also true for any even k.

Let us consider the case $k = 3$ corresponding to the spin $j = 1/2$. In this case the supercharges $Q^\pm$ acquire the form

$$Q^\pm = 2^{-\frac{3}{2}} A^\mp \left(\left(A^\mp\right)^2 - 1 \right) \theta^\pm .$$

The zero modes of the operator Q^+ are

$$\varphi_1 = e^{-a\cosh x}, \qquad \varphi_{2,3} = e^{\pm x} e^{-a\cosh x}.$$

Looking for the solutions to the equation (4.9) in the form of the linear combination of the zero modes, $\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$, we obtain the following three (not normalized) eigenfunctions

$$\varphi_0 = \sinh x \, e^{-a\cosh x}, \qquad \varphi_\pm = (1 + c_\pm \cosh x)\, e^{-a\cosh x},$$

where $c_\pm = (4a)^{-1} \cdot (1 \mp \sqrt{16a^2+1})$. The energies of these states are

$$E_0 = -\frac{1}{2}, \qquad E_\pm = \frac{1}{4}\left(\pm \sqrt{16a^2+1} - 1 \right),$$

and the anticommutator of the supercharges is

$$\left\{Q^-, Q^+\right\} = (H - E_0)\,(H - E_-)\,(H - E_+)\,. \tag{4.10}$$

It is necessary to stress that though in the case $k = 3$ the direct application of the partial algebraization scheme to the potential allows ones to find only two eigenstates and corresponding eigenvalues ($\varphi_\pm$, $E_\pm$), the concept of nonlinear supersymmetry gives the information on one more exact eigenstate (φ_0) of the Hamiltonian. The same is also valid for any odd k: the algebraization scheme gives for the potential (4.7) the information on $\frac{k+1}{2}$ eigenvalues corresponding to the even (in x) eigenstates, whereas the nonlinear supersymmetry allows ones to find in addition $\frac{k-1}{2}$ eigenvalues corresponding to the odd eigenfunctions.

The eigenfunctions and eigenvalues in the fermionic sector for $a < 0$ can be obtained via the formal change $a \to -a$, and we conclude that in the case $k = 3$ there are three bound states in bosonic (fermionic) sector for $a > 0$ ($a < 0$).

Now let us generalize the k-supersymmetry of the system given by the superpotential (4.8) for the case of arbitrary k. Using the explicit form of the supercharges for $k = 2, 3$, one can suppose that the supercharges $Q_k^\pm$ obey the recurrent relation

$$Q_{k+2}^\pm = \frac{1}{2}\left(\left(A^\mp\right)^2 - \left(\tfrac{k+1}{2}\right)^2 \right) Q_k^\pm . \tag{4.11}$$

Let us prove the conservation of the supercharges (4.11) by the method of mathematical induction using this conjecture. As the first step we suppose that for some given k the supercharge is conserved. Thus, calculating $[Q_k^+,\, H_k]$, we obtain the relation

$$\left[W', Q_k^+\right] + kWA^- Q_k^+ + \frac{1}{4}\left(k-1\right) kW' Q_{k-2}^+ = 0. \tag{4.12}$$

The Hamiltonian H_{k+2} is related to the H_k as $H_{k+2} = H_k - W' + 2W'N$. Then we have

$$\left[Q_{k+2}^+,\, H_{k+2}\right] = \left[W',\, Q_{k+2}^+\right] + \left(k+2\right) WA^- Q_{k+2}^+ + \frac{1}{4}\left(k+1\right)\left(k+2\right) W' Q_k^+.$$

Using the relations (4.11) and (4.12), after rather cumbersome algebraic manipulations one can reveal that the expression vanishes. Since the corresponding supercharges are conserved for $k = 2,\, 3$, we conclude that the supercharges $Q_k^\pm$ are conserved for any k as well. This also proves the conjecture (4.11).

The relation (4.11) gives us the following form of the supercharges for arbitrary k:

$$Q_k^\pm = 2^{-\frac{k}{2}}\left(A^\mp + \tfrac{k-1}{2}\right)\left(A^\mp + \tfrac{k-3}{2}\right)\ldots\left(A^\mp - \tfrac{k-3}{2}\right)\left(A^\mp - \tfrac{k-1}{2}\right)\theta^\pm.$$

Using this representation, we find all the zero modes of the supercharge Q_k^+:

$$\varphi_s(x) = e^{sx - a\cosh x}, \qquad s = -\tfrac{k-1}{2},\, -\tfrac{k-3}{2},\, \ldots,\, \tfrac{k-3}{2},\, \tfrac{k-1}{2}.$$

Since these modes belong to the bosonic sector, we can build k eigenfunctions of the bosonic part of the Hamiltonian as a linear combination of them:

$$\psi(x) = e^{-a\cosh x}\begin{cases} \displaystyle\sum_{n=-m}^{m} c_n e^{nx}, & k = 2m+1, \\ \displaystyle\sum_{n=-m}^{m-1} c_n e^{\left(n+\frac{1}{2}\right)x}, & k = 2m. \end{cases}$$

If we put this into the corresponding stationary Schrödinger equation, we arrive at the recurrent system of algebraic equations on energy E and the coefficients c_n

$$\frac{1}{a}\left(n^2 + 2E\right)c_n + c_{n+1}\left(m+n+1\right) + c_{n-1}\left(m-n+1\right) = 0, \qquad \text{for odd } k,$$

where $c_n = 0$ for $|n| > m$, and

$$\frac{1}{a}\left(\left(n+\tfrac{1}{2}\right)^2 + 2E\right)c_n + c_{n+1}\left(m+n+1\right) + c_{n-1}\left(m-n\right) = 0, \qquad \text{for even } k,$$

where $c_n = 0$ for $n < -m$ or $n \geq m$. Thus, we have demonstrated how the concept of nonlinear supersymmetry allows us to reduce the problem of finding the part of the spectrum for the potential (4.7) to the pure algebraic problem.

Let us discuss shortly the potential (4.4), to which the superpotential $W(x) = \frac{a}{2}\sinh 2x$ corresponds within the framework of the nonlinear supersymmetry. From the form of the superpotential one can immediately conclude that the potential (4.4) can be reduced to the

form (4.7) just by rescaling the argument and the parameter a. Indeed, with the help of the relation

$$\frac{a^2}{2}\cosh^4 x - \frac{a}{2}(a+4j+2)\cosh^2 x = \frac{a^2}{8}\sinh^2 2x - \frac{2j+1}{2}a\cosh 2x - \frac{2j+1}{2}a,$$

after rescaling $x \to 2x$, $a \to a/4$, we arrive at the form (4.3). But in the case of the potential (4.4), there is a difference in comparison with (4.3). As we have mentioned above, according to Ref. [21, 34], only the even eigenstates of Hamiltonian with the potential (4.3) can be found following the $sl(2,\mathbb{R})$ partial algebraization scheme, whereas for the potential (4.4) all the lowest $2j+1$ states can be found within the framework of the very scheme. But we see that though the potentials (4.3) and (4.4) have different representations in the algebraization scheme, in reality they correspond to the same physical system. Moreover, we see that all the potentials (4.1)-(4.4) are particular cases of the potential (3.5) with the exponential superpotential (3.11). For the potential (3.5) the first k states can be found algebraically with the help of the associated nonlinear supersymmetry. In other words, the nonlinear supersymmetry provides us with a universal point of view on the quasi-exactly solvable potentials (4.1)-(4.4).

Now let us discuss the superpotential (3.11) of the general form, which, as we have seen, allows ones to unify all the cases (4.1)-(4.4). The potential (3.5) with this superpotential has the form

$$V(x) = \frac{1}{2}w_+^2 e^{2\omega x} + \left(w_0 - \tfrac{k}{2}\omega\right) w_+ e^{\omega x} + \left(w_0 + \tfrac{k}{2}\omega\right) w_- e^{-\omega x} + \frac{1}{2}w_-^2 e^{-2\omega x} + v + w_+ w_- + \frac{1}{2}w_0.$$

The superpartner of this potential can be obtained by the formal substitution $k \to -k$. Consider in detail the simplest case $k=2$. The corresponding supercharge

$$Q^+ = \frac{1}{2}\left((A^-)^2 - \frac{\omega^2}{4}\right)\theta^+$$

has the zero modes

$$\varphi_\pm = \exp\left(\left(\pm\tfrac{\omega}{2} - w_0\right)x - \tfrac{1}{\omega}\left(w_+ e^{\omega x} - w_- e^{-\omega x}\right)\right), \tag{4.13}$$

and for the equation $H_B\varphi = E\varphi$ one finds the two eigenfunctions

$$\varphi^{(\pm)} = \varphi_- + c_\pm \varphi_+, \qquad c_\pm = \frac{1}{2w_-}\left(w_0 \pm \sqrt{w_0^2 - 4w_+w_-}\right),$$

with the energy

$$E_\pm = \pm\frac{1}{2}\omega\sqrt{w_0^2 - 4w_+w_-} + v - \frac{\omega^2}{8}.$$

In the fermionic sector, the zero modes of the supercharge Q^- are

$$\psi_\pm = \exp\left(\left(\pm\tfrac{\omega}{2} + w_0\right)x + \tfrac{1}{\omega}\left(w_+ e^{\omega x} - w_- e^{-\omega x}\right)\right) \tag{4.14}$$

and all the corresponding formulas can by reproduced via the changes $w_\pm \to -w_\pm$, $w_0 \to -w_0$. As it follows from (4.13) and (4.14), in the case $w_+w_- < 0$ there are two bound states

in bosonic (fermionic) sector when $w_+ > 0$ ($w_+ < 0$), whereas for $w_+ w_- > 0$ there are no such bound states at all. The corresponding $k = 2$ superalgebra is written as

$$\left\{Q^-, Q^+\right\} = \left(H - \tfrac{1}{2}q\right)^2 + q\left(w_0^2 - 4w_+ w_-\right).$$

Using the obtained results for the potential (4.7) and realizing the substitution $x \to \omega x$, we arrive at the supercharges obeying the recurrent relation

$$Q^\pm_{k+2} = \frac{1}{2}\left(\left(A^\mp\right)^2 - \left(\tfrac{k+1}{2}\right)^2 \omega^2\right) Q^\pm_k.$$

From the consideration of the nonlinear supersymmetry with the superpotential (3.11), it follows that for any k the supercharges depend on the parameter ω_k and do not depend explicitly on other parameters of the superpotential. This means that the supercharges are conserved and obey the same recurrent relation if we rescale the parameter q to yield the relation $\omega_k = \omega$. Then in the same way as for the potential (4.7), one can look for the wave functions of the k singlets in the form

$$\psi(x) = e^{-\int^x W(y)\,dy} \begin{cases} \displaystyle\sum_{n=-m}^{m} c_n e^{n\omega x}, & k = 2m+1, \\ \displaystyle\sum_{n=-m}^{m-1} c_n e^{\left(n+\frac{1}{2}\right)\omega x}, & k = 2m. \end{cases}$$

The corresponding stationary Schrödinger equation gives us the recurrent system of algebraic equations on the energy and on the coefficients c_n:

$$c_n(E - v + \tfrac{1}{2}n\omega(n\omega - 2w_0)) + (m - n + 1)\omega w_+ c_{n-1} - (m + n + 1)\omega w_- c_{1+n} = 0,$$

for odd k, where $c_n = 0$ for $|n| > m$, and

$$c_n(E - v + \tfrac{1}{2}(\tfrac{1}{2} + n)\omega((\tfrac{1}{2} + n)\omega - 2w_0)) + (m - n)\omega w_+ c_{n-1} - (m + n + 1)\omega w_- c_{n+1} = 0,$$

for even k, where $c_n = 0$ for $n < -m$ or $n \geq m$. Excluding the coefficients c_n from the recurrent systems, one can obtain the algebraic equation for E of the order k. Normalizing the corresponding polynomial to the form $E^k + \ldots$ and substituting $E \to H$, one can get the exact form of the polynomial specifying the k-supersymmetry via the anticommutator of the supercharges $Q^\pm$. Similarly to the case $k = 2$, one can find that if $w_+ w_- < 0$, then there are k bound states in bosonic (fermionic) sector when $w_+ > 0$ ($w_+ < 0$). If $w_+ w_- > 0$, such bound singlets do not exist at all.

To conclude this section, we note that the choice of the parameters in the superpotential (3.11)

$$\omega_k = \alpha, \quad w_+ = 0, \quad w_- = \pm B, \quad w_0 = \mp A - \frac{\alpha}{2}(k \pm 1),$$
$$v = -\frac{1}{8}\alpha(k \pm 1)((k \pm 1)\alpha \pm 4A)$$

leads to the exactly solvable Morse potential [4]

$$V(x) = A^2 + B^2 e^{-2\alpha x} - B(2A + \alpha)e^{-\alpha x}, \tag{4.15}$$

where A, B are real positive parameters. As it is well known, from the Morse potential a vast number of famous exactly solvable potentials can be reproduced by the point canonical transformations and suitable limiting procedures [4]. This illustrates an intimate though indirect relation of the nonlinear supersymmetry not only to the quasi-exactly solvable systems but to the exactly solvable models as well.

5 Anomaly-free $k = 2$ supersymmetry

In the subsection 2.3, we have analysed the classical supersymmetry from the viewpoint of the polynomial in momentum structure of the supercharges. Since for the second order supercharges the solution was found by us in the general form, let us analyse this case at the quantum level too. On the other hand, there is no sense to analyse in such a context the orders higher then 2 since we do not know the general solution even at classical level. Different aspects of supersymmetry with the general second order supercharges were considered in Ref. [30]. Here we discuss the general quantum case of the $k = 2$ supersymmetry in the context of "curing" the quantum anomaly problem and from the viewpoint of all the possible types of spectra. Then, as an example of application of the $k = 2$ supersymmetry, we construct a new exactly solvable nontrivial quantum model associated with the infinitely deep well.

5.1 Quantum $k = 2$ supersymmetry vs. classical supersymmetry

Let us consider the second order supercharge of the general form

$$Q^+ = \frac{1}{2}\left(\hbar^2 \frac{d^2}{dx^2} + 2\hbar f(x)\frac{d}{dx} + b(x)\right)\theta^+, \tag{5.1}$$

and the most general quantum Hamiltonian

$$H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x) + L(x)\theta^+\theta^-.$$

The condition $[Q^+, H] = 0$ partially determines the unknown functions in Q and H:

$$L(x) = 2f'(x), \tag{5.2}$$

$$V(x) = \frac{1}{2}\left(f - \frac{\hbar f'}{2f}\right)^2 - \frac{\hbar}{2}\left(f - \frac{\hbar f'}{2f}\right)' - \frac{c}{2f^2} + v, \tag{5.3}$$

$$b(x) = \frac{c}{f^2} + f^2 + \hbar f' - \frac{\hbar^2 f''}{2f} + \left(\frac{\hbar f'}{2f}\right)^2, \tag{5.4}$$

where c and v are real constants. The function (5.3) plays the role of the potential for the bosonic sector. Taking into account Eqs. (5.2)–(5.4) and identifying $f(x) = W(x)$, we obtain

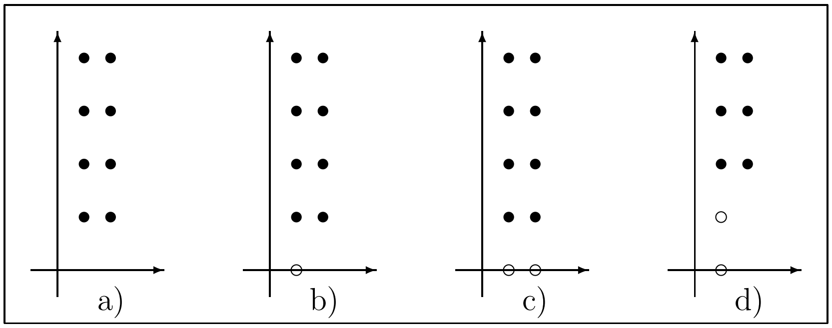

Figure 1: The four types of the spectra for the $k = 2$ supersymmetry.

the following most general form for the Hamiltonian and the supercharge of the quantum $k = 2$ supersymmetry:

$$H = \frac{1}{2}\left[-\hbar^2\frac{d^2}{dx^2} + W^2 - \frac{c}{W^2} + 2v + 2\hbar W'\sigma_3 + \Delta(W)\right] \tag{5.5}$$

$$Q^+ = \frac{1}{2}\left[\left(\hbar\frac{d}{dx} + W\right)^2 + \frac{c}{W^2} - \Delta(W)\right]\theta^+, \tag{5.6}$$

$$\Delta = \frac{\hbar^2}{4W^2}\left(2W''W - W'^2\right). \tag{5.7}$$

Looking at the quantum Hamiltonian (5.5) and supercharge (5.6), we see that their form is different from the direct quantum analogue constructed proceeding from the classical quantities (2.29) and (2.28) via the quantization prescription $N = \frac{1}{2}[\theta^+, \theta^-]$: the presence of quadratic in $\hbar^2$ term (5.7) in both operators H and Q^+ is crucial for preserving the supersymmetry at the quantum level. Therefore, one can say that the quantum correction (5.7) cures the problem of the quantum anomaly since without it the supercharge would not be the integral of motion. It is interesting to note that the quantum term of the form (5.7) appears also in the method of constructing new solvable potentials from the old ones via the operator transformations [4].

The supercharges Q^+ and $Q^- = (Q^+)^\dagger$ satisfy the relation

$$\left\{Q^+, Q^-\right\} = (H - v)^2 + c \tag{5.8}$$

being the exact quantum analogue of the first classical relation from Eq. (2.30). This superalgebra defines the form of the spectrum of bounded states. For the sake of definiteness, let us put the constant $v = 0$. Then in general the spectra of the $k = 2$ supersymmetric system can be of the four types presented on Fig. 1. For $c > 0$, there are no singlets in the system and we have completely broken $k = 2$ SUSY, see Fig. 1**a**. This case corresponds, in particular, to the $k = 2$ SUSY given by the quadratic superpotential discussed in Section 3.1.

In the case $c = 0$, there are three possibilities: **(i)** the completely broken phase (there are no singlet states), see Fig. 1**a**; **(ii)** there is one singlet state in either bosonic or fermionic sector, see Fig. 1**b**; **(iii)** there are two singlet states with equal energy, one in bosonic and another in fermionic sector, see Fig. 1**c**. The types of the spectra **b** and **c** are represented by the new exactly solvable model in the next subsection.

In the case $c < 0$, all the three mentioned types of the spectra, **a**, **b** and **c**, can be realized and, in addition, another situation with two singlet states in one of the two (bosonic or fermionic) sectors can exist, see Fig. 1**d**. Examples of the type **d** were represented above exhaustively by the supersymmetric systems associated with the exponential superpotential (3.11).

5.2 The $k = 2$ supersymmetry in action: a new exactly solvable model

Via the appropriate choice of the superpotential, one can construct the $k = 2$ supersymmetric systems associated with the exactly solvable potentials. E.g., the simplest choice of the superpotential $W(x) = x$ leads to the famous (two-particle) Calogero model. The Morse potential (4.15) can also be reproduced and there are two possibilities to realize it here:

$$W(x) = A - \frac{1}{2}\alpha - Be^{-\alpha x}, \qquad\qquad c = -\frac{1}{16}\alpha^2 \left(2A - \alpha\right)^2,$$

or

$$W(x) = Be^{-\alpha x} - A - \frac{3}{2}\alpha, \qquad\qquad c = -\frac{1}{16}\alpha^2 \left(2A + 3\alpha\right)^2.$$

It is well-known that the usual linear supersymmetry allows ones to obtain new exactly solvable potentials from the given ones. Here we show that the $k = 2$ supersymmetry can be exploited in the same way. For example, one can find that in the case $c \leq 0$ the potential (5.3) has exactly the form of the potential of the linear supersymmetry with the superpotential

$$\mathcal{W}(x) = f(x) - \frac{f'(x) - 2\sqrt{-c}}{2f(x)}.$$

Here and in what follows we put $\hbar = 1$. If the superpotential $\mathcal{W}(x)$ is given, we can consider this relation as differential equation. Solving this equation, we obtain the one-parametric solution for $k = 2$ superpotential $f(x) = W(x)$. Though the initial potential is the same for linear ($k = 1$) and $k = 2$ supersymmetric systems, the corresponding superpartners are different: $V_2^{k=1} = V + \mathcal{W}'$, while $V_2^{k=2} = V + 2f'$. Therefore, in the case $k = 2$ one can construct the one-parametric family of isospectral potentials as superpartner to $V(x)$.

We will illustrate this by the example of the infinite square potential well. Without loss of generality, we can assume that the width of the potential is equal to 1. This problem is equivalent to the equation

$$\psi''(x) + (2E - \pi^2)\psi(x) = 0$$

with the boundary conditions

$$\psi(0) = \psi(1) = 0,$$

and we can think that x runs over the segment $[0,\ 1]$ only. The eigenfunctions have the form

$$\psi_n(x) = \sqrt{2}\sin(n+1)x \tag{5.9}$$

with the energy

$$E_n = \frac{\pi^2}{2} n(n+2), \tag{5.10}$$

where the energy of the ground state has been chosen to be equal to 0.

Let us consider the infinite square potential well as a bosonic potential of the system with the $k = 2$ SUSY. In order to obtain the superpartner of the potential, we have to find the superpotential $f(x)$. For simplicity we put $c = 0$. Then to find the superpotential, we have to solve the equation

$$f(x) - \frac{f'(x)}{2f(x)} = -\frac{\psi_0'(x)}{\psi_0(x)}.$$

The general solution has the form

$$f(x) = \frac{2\pi \sin^2 \pi x}{c_0 - 2\pi x + \sin 2\pi x}. \tag{5.11}$$

In order the superpotential to be well defined function on (0, 1), we have to assume that the real constant c_0 can take any value in $\mathbb{R}$ except the interval $(0, 2\pi)$. Let us note that the superpartner in the framework of the usual ($k = 1$) supersymmetry is proportional to $\operatorname{cosec}^2 \pi x$ and has a pure trigonometric nature, while this is not the case for the $k = 2$ SUSY superpartner.

The superpartner of the potential is defined as $V_f(x) = V(x) + 2f'(x)$, and in this case it acquires the form

$$V_f(x) = -\frac{\pi^2}{2} + 16\pi^2 \frac{\sin \pi x \left(\sin \pi x - \left(\pi x - \frac{1}{2} c_0\right) \cos \pi x\right)}{\left(c_0 - 2\pi x + \sin 2\pi x\right)^2}. \tag{5.12}$$

Acting by the supercharge Q^+ of the form (5.1) with the superpotential (5.11) on the wave functions (5.9) of the square potential well, we obtain the eigenfunctions of the potential (5.12),

$$\psi_n^{(f)}(x) = \mathcal{N} \left(\frac{4\,(1+n) \sin \pi x \sin n\pi x}{c_0 - 2\pi x + \sin 2\pi x} + \left(n^2 + \frac{2n\,(c_0 - 2\pi x)}{c_0 - 2\pi x + \sin 2\pi x} \right) \sin(1+n)\pi x \right),$$

given here up to a normalization constant $\mathcal{N}$. One can verify that $\psi_0^{(f)}(x)$ is identically equal to zero.

It is worth noting that the function $\psi_n^{(f)}(x)$ has n nodes when $c_0 \neq 0, 2\pi$ and $n-1$ nodes when $c_0 = 0$ or $c_0 = 2\pi$. This means that in the latter case $\psi_1^{(f)}(x)$ is the ground state of the potential (5.12), but this is not valid for the former case. This situation is illustrated on Figs. 2, 3 by the example of the function $\psi_1^{(f)}(x)$. The arrows indicate direction of the node motion when c_0 goes to 0 (Fig. 2) or to 2π (Fig. 3). For the case $c_0 \neq 0, 2\pi$, the ground state can be found by calculating the limit $\lim_{n\to 0} \psi_n^{(f)}(x)/n$. Having done this, one can verify that the function obtained in such a way,

$$\psi_0(x) = \sqrt{2c_0\,(c_0 - 2\pi)} \frac{\sin \pi x}{c_0 - 2\pi x + \sin 2\pi x}, \qquad \|\psi_0\| = 1,$$

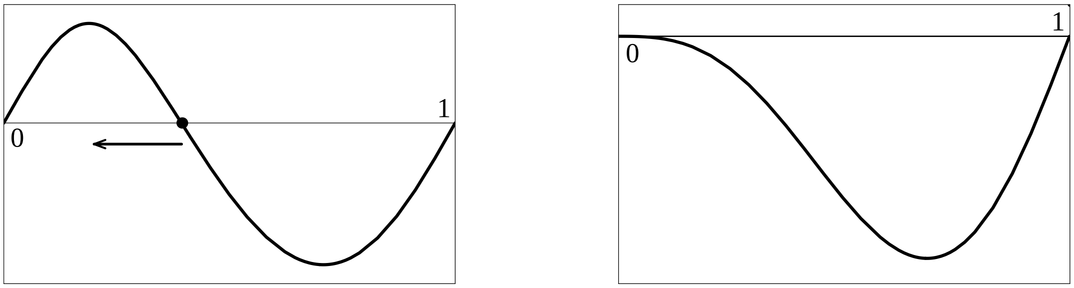

Figure 2: The plots of the function $\psi_1^{(f)}(x)$ for the cases $c_0 < 0$ and $c_0 = 0$.

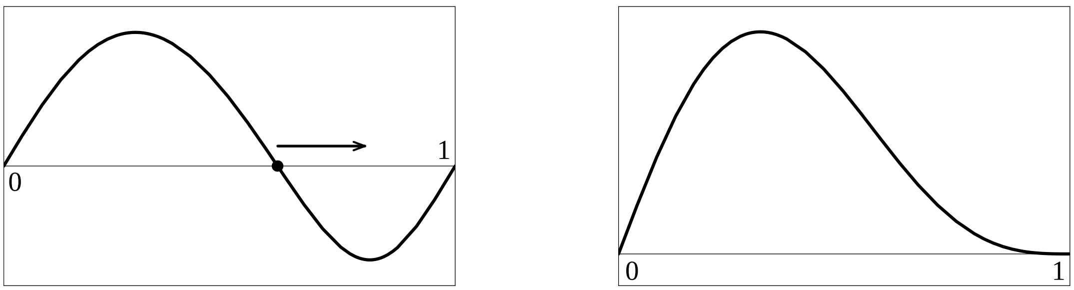

Figure 3: The plots of the function $\psi_1^{(f)}(x)$ for the cases $c_0 > 2\pi$ and $c_0 = 2\pi$.

is indeed the ground state of the potential. The function vanishes when $c_0 = 0$ or $c_0 = 2\pi$ that corresponds to the statement above. Therefore, by a choice of the parameter c_0 we can obtain the unbroken $k = 2$ SUSY of the two types, namely, with one singlet for $c_0 = 0$ or $c_0 = 2\pi$, and with two singlets with equal energies for other admissible values of c_0.

Thus, we have illustrated the special case of the $k = 2$ supersymmetric system by the example of the infinite square potential well. We have also shown that in the case $c = 0$, the two types of the spectra represented on Fig. 1**b** and **c** can be realized. The spectrum of the first type has exactly the form of that of a system with the usual supersymmetry in the exact phase. The spectrum of the second type has rather unusual properties. It has the form characteristic to a system with $k = 1$ broken supersymmetry (with coinciding two lowest energy levels), but here the two lowest states with equal energies are the true supersymmetric singlets and, therefore, the $k = 2$ supersymmetry is unbroken.

6 Discussion and outlook

Let us summarize briefly the obtained results and then discuss some open problems that deserve further attention.

- Classical supersymmetry is characterized by the Poisson algebra being nonlinear in the Hamiltonian, and includes the usual linear supersymmetry as a particular case.

- Any supersymmetric system is symplectomorphic to the supersymmetric system of the canonical form with the holomorphic supercharges.

- The canonical supersymmetric systems are separated into the three classes defined by the behaviour of the superpotential. In the first class the parameter α characterizing the order of the superalgebra is subject to a classical quantization: $\alpha = k \in \mathbb{Z}_+$; in the second class it can take any non-negative value: $\alpha \in \mathbb{R}_+$; in the systems of the third class the fermion degrees of freedom can be decoupled completely by a canonical transformation and, hence, their supersymmetry has a rather fictive nature.

- We have investigated the nonlinear supersymmetry with supercharges being polynomial of order k in the momentum, and for $k = 2$ have found the most general one-parametric solution of the Calogero-like form (2.29). Depending on the value of the parameter c, classically the Calogero-like $k = 2$ supersymmetric system can be reduced to the $k = 0$, $k = 1$ or $k = 2$ supersymmetric system of a canonical form.

The quantization of a system with nonlinear supersymmetry is a nontrivial problem due to the quantum anomaly taking place in general case. We have shown that

- The anomaly-free quantization of the classical k-supersymmetric system with the holomorphic supercharges is possible for the superpotential of the quadratic (3.8) and exponential forms (3.11) only;

- The k-supersymmetric systems with the exponential superpotential are closely related to the well-known families of the quasi-exactly solvable systems [21];

- The $k = 2$ supersymmetric Calogero-like systems can be quantized in the anomaly-free way. The problem of the quantum anomaly is "cured" here by the specific superpotential-dependent term of order $\hbar^2$. Such a quantum term appeared earlier in the operator transformations method of constructing new solvable potentials from the known ones [4];

- The general $k = 2$ supersymmetry associated with the Calogero-like systems can be used for producing new exactly solvable potentials.

The supersymmetric systems (2.21) given by the superpotentials of the third type are canonically equivalent to the system (2.20) with the completely decoupled fermion degrees of freedom. On the other hand, the $k = 1$ quantum analogues of (2.21) with superpotential of the third type are the systems with spontaneously broken linear supersymmetry. In this case the nilpotent operators $\tilde{Q}^{\pm} = Q^{\pm}/\sqrt{2H}$ are well defined and their anticommutator is equal to 1, i.e. $\tilde{Q}^{\pm}$ look like $k = 0$ quantum supercharges. However, these operators are not decoupled from the even operators x and p. Therefore, the question is whether the quantum analogue of the above mentioned canonical transformations exists. If so, the quantum fermion degrees of freedom could be completely decoupled and the initial $k = 1$ supersymmetric system would be reduced to the $k = 0$ supersymmetric system in correspondence with the classical picture. On the other hand, if such a transformation does not exist (at least for some superpotentials $W(x)$), we face a sort of quantum transmutation. The classical equivalence of the Calogero-like $k = 2$ supersymmetric systems to the $k = 0$ (for $c > 0$) and to the $k = 1$ (for $c < 0$) supersymmetric systems of the canonical form (2.21) is also based on the existence of the corresponding canonical transformations. If for the quantum Calogero-like

$k = 2$ supersymmetric systems (5.5)–(5.7) the quantum analogues of the above mentioned canonical transformations do not exist, we again have a quantum transmutation.

The supersymmetric system (2.21) represents the whole class of the symplectomorphic systems (2.15). However, the quantization apparently breaks the equivalence of the systems (2.15) with different functions $M(W^2)$. Therefore the quantization of the systems (2.15) may lead to different nontrivial quantum systems with k-supersymmetry. Most likely, the anomaly-free quantization could be possible only for some special cases of the function $M(W^2)$ and the superpotential $W(x)$. In this context it is necessary to stress that the quantum anomaly is specific not only for the nonlinear supersymmetry. The quantization of the general system (2.15) with $k = 1$ and $M(W^2) \not\equiv 0$ gives rise to the quantum anomaly. From this view-point one can say that the system of the form (2.21) underlies the quantum linear supersymmetry since the quantization of such a system with arbitrary superpotential does not reveal any anomaly. However, it would be interesting to find at least one example of the quantum-mechanical system of the non-standard form (2.15) with the linear supersymmetry. Since for the nontrivial cases the quantum analogue of the transformation (2.13) is non-local, the corresponding system has to operate with non-local objects. It is possible that the supersymmetry of the pure parabosonic systems [10] can be arrived at in this way. We hope that further investigations will shed light on the relation between the classical and quantum supersymmetries in the context on the described hypothetical quantum transmutations and the quantum anomaly problem.

Though we have considered the concept of the nonlinear supersymmetry for one-dimensional systems, it can be extended to the higher dimensional systems as well. For example, the two-dimensional system describing the motion of a charged particle in external magnetic field admits the nonlinear supersymmetries. The Pauli Hamiltonian of such a system is given by ($\hbar = m = e = 1$) $H = \frac{1}{2}(\mathcal{P}_1^2 + \mathcal{P}_2^2 + g\varepsilon_{ij}\partial_i A_j N)$, where $\mathcal{P}_i = -i\partial_i + A_i$, and A_i, $i = 1, 2$, form the 2D vector gauge potential. For the gyromagnetic ratio $g = 2$ this system reveals the usual linear supersymmetry both at the classical and quantum levels [4]. It turns out that for $g = 2k$ the system possesses the k-supersymmetry. To show this let us introduce the complex variables $\tilde{z}^{\pm} = \mathcal{P}_1 \pm i\mathcal{P}_2$. In terms of these variables the Hamiltonian can be rewritten as $H = \frac{1}{2}(\tilde{z}^+\tilde{z}^- + ik\{\tilde{z}^+, \tilde{z}^-\}N)$. But this is exactly the form of the Hamiltonian (2.21) with the variables z, $\bar{z}$ changed for $\tilde{z}^+$, $\tilde{z}^-$. Therefore, the Hamiltonian commutes with the supercharges $Q^{\pm} = (\tilde{z}^{\mp})^k\theta^{\pm}$ and, as a consequence, the system possesses the classical k-supersymmetry. The quantization of this system reveals the properties similar to those for the k-supersymmetric system with the holomorphic supercharges: the requirement of conservation of the nonlinear supersymmetry leads to the restrictions on the form of the magnetic field $B = \varepsilon_{ij}\partial_i A_j$. One can show that as in the one-dimensional case, the form of the magnetic field turns out to be restricted by the quadratic case and by the sum of exponential functions in the variables x_i [35]. We suppose that such systems are also closely related to the two-dimensional quasi-exactly solvable problems.

To conclude, we note that our investigation of the nonlinear supersymmetry was motivated by the existence of nonlinear supersymmetry in pure parabosonic systems [10], where the reflection (parity) operator R plays the role of the Z_2-grading operator. Such a special role of the reflection operator was the main idea in the construction of the minimally bosonized $k = 1$ supersymmetric quantum mechanics [36, 37], where the role of the superpotential is played by the arbitrary odd function. Therefore, the construction of the bosonized

nonlinear supersymmetry is an attractive problem which we hope to consider elsewhere [35].

When this paper was finished, the interesting papers [38, 39] devoted to the development of the $1D$ nonlinear supersymmetry of ref. [33] have appeared. We note that the generalization of the k-supersymmetry with the holomorphic supercharges found in ref. [39] and possessing the typical Calogero-like structure of the $k = 2$ supersymmetry from Section 5 could also be treated in the context of the quasi-exactly solvable systems discussed here.

Acknowledgements

The work was supported by the grants 1980619, 1010073 and 3000006 from FONDECYT (Chile) and by DYCIT (USACH).

Appendix

Here we show that the arbitrariness in the definition of the superpotential, $\tilde{W}^2(x) = W^2(x)+ const$, can not be used to change the type of the classical supersymmetry of the given system, i.e. we demonstrate the invariance of the classification obtained in Section 2.2. For the purpose, it is enough to prove that the redefinition of the superpotential can not provoke a transition between the adjacent classes of supersymmetry.

Let us start by analysing the possibility of the transition from the supersymmetry of the first type with the superpotential unbounded from below to the supersymmetry of the second type. For the sake of simplicity we assume that the potential $V(x)$ has finite number of minima, the lowest one is at the origin and is equal to zero. For such a potential, the superpotentials of the first and second types obey the relation $W^2(x) = V(x)$. Suppose that the bounded and unbounded superpotentials $W_u(x)$ and $W_b(x)$ with the associated supersymmetries of the first and second types correspond to the potential. The superpotential $W_b(x)$ is a continuously differentiable function at $x = 0$ only when the potential has the local behaviour $V(x) = x^4 + O(x^5)$. Indeed, if $V(x) = x^2 + O(x^3)$, the bounded superpotential locally is $W_b(x) = |x| + O(x^2)$, and as a consequence, its derivative is not regular at the origin. The case of the potential admitting the regular superpotential $W_b(x)$ is presented on the Fig. 4. In this case the bounded and unbounded superpotentials $W_b(x)$ and $W_u(x)$ coincide for positive values of x and have different sign for negative x: $W_u(x) = \mathrm{sign}(x)W_b(x)$. If we have the nonlinear supersymmetry of the first type, i.e. $L(x) = kW_u'(x)$, $k \in \mathbb{N}$, then rewriting this function in terms of W_b we obtain $L(x) = k\,\mathrm{sign}(x)W_b'(x)$. The function $L(x)$ is not good in terms of W_b in the sense that there exist no canonical transformation that could reduce this system to that with the supersymmetry of the second type given by Eq. (2.18). The potential with several minima can be considered in a similar way.

The case of supersymmetry of the first type with a bounded superpotential merits the special analysis. After the transition (2.22) to the superpotential $\tilde{W}$ of the second type, the function $L(x)$ can be rewritten as $L(x) = k\tilde{W}'\tilde{W}/(\tilde{W}^2 - \tilde{w})^{1/2}$, $\tilde{w} = const$. The function $(\tilde{W}^2 - \tilde{w})^{-1/2}$ has singularities at the points where the initial superpotential vanishes. Therefore, the type of the supersymmetry can not be changed either.

Now let us consider the possibility of transition from the supersymmetry of the second type to that of the third type by means of the change (2.22). We suppose that the

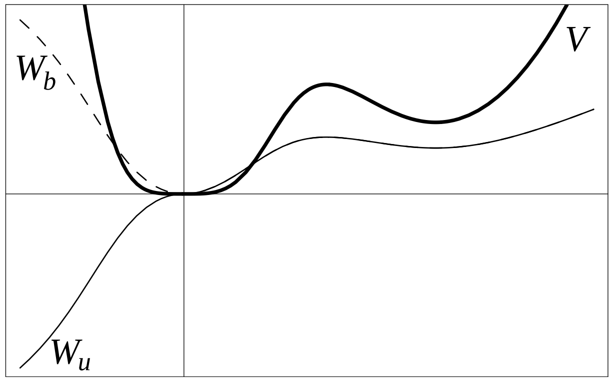

Figure 4: The plots of a potential $V(x)$ and the corresponding superpotentials $W_u(x)$ and $W_b(x)$.

superpotential $W(x)$ of a given system has a minimum equal to zero at the origin of coordinates. The superpotential of the third type defined by $W^2(x) = \tilde{W}^2(x) - w$ has the minimum equal to w at the origin. Rewriting $L(x)$ in terms of the new superpotential, we obtain $L(x) = \alpha W' = \alpha \tilde{W}' \tilde{W}/(\tilde{W}^2 - w)^{1/2}$. There exists no canonical transformation (2.13) that could remove the corresponding nilpotent part of the Hamiltonian since the function $\tilde{W}/(\tilde{W}^2 - w)^{1/2}$ is singular at the origin. The case of the superpotential with several minima can be treated in a similar way. This completes our proof of the invariance of the obtained classification of the classical supersymmetries.

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